On the use of the Cauchy integral formula

- Motivation
- Cauchy & Biot and Savart
- Surface current method
- Example
    - Dipole
    - Quadrupole
    - undulator

Augustin-Louis Cauchy
1789-1857

INTRODUCTORY REMARK: 18TH -19TH CENTURY

Mathematical integration in physics

Newton

Euler, Franklin — 1750

The foundation of electromagnetism:

Lagrange

Volta, Laplace, Coulomb

Fourier, Legendre — 1800

Ampere, Faraday Ohm, Poisson, Oersted

Henry, Cauchy, Green, Biot

Gauss, Lenz, Helmotz , Joule

Stokes, Kelvin, Riemann — 1850

Maxwell

Cantor

Boltzman

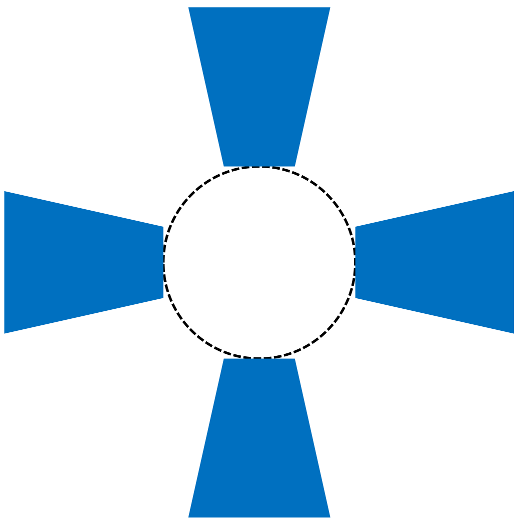
Becquerel, Lorentz , Hertz — 1900

Einstein, Curie

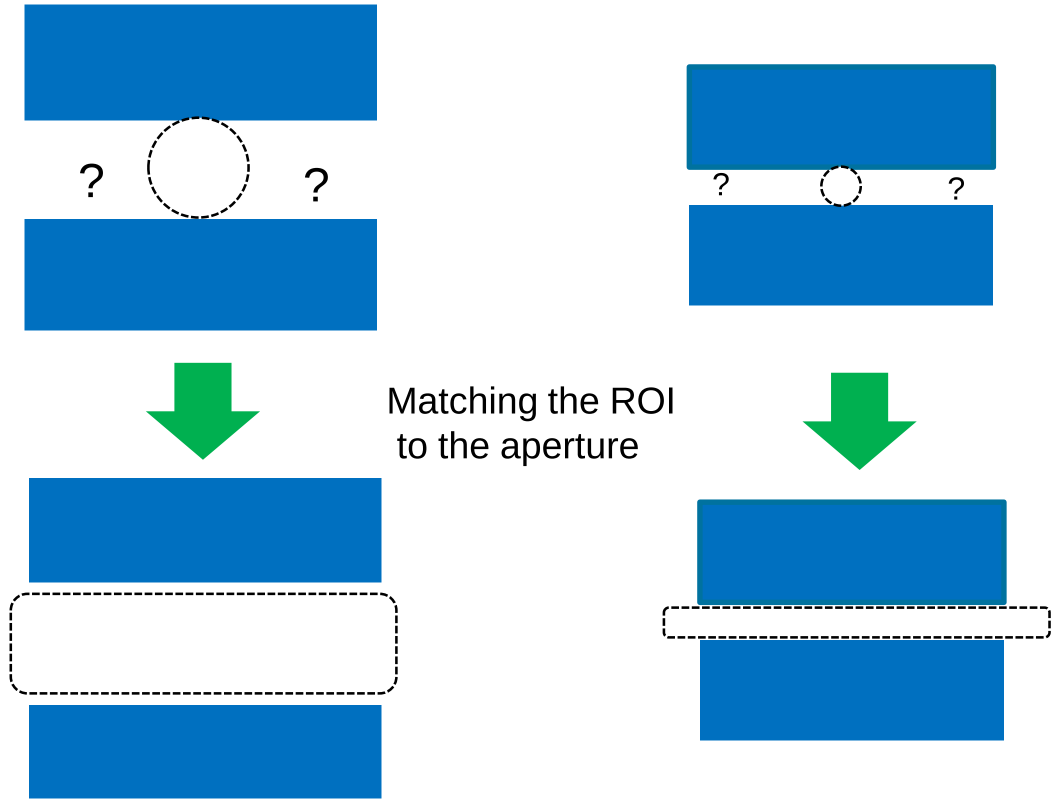
FIELD INTEGRAL MEASUREMENTS

Potential difficulty with measurement in apertures with large aspect ratio
Example with usual circular measurement

Circular bore radius



OK



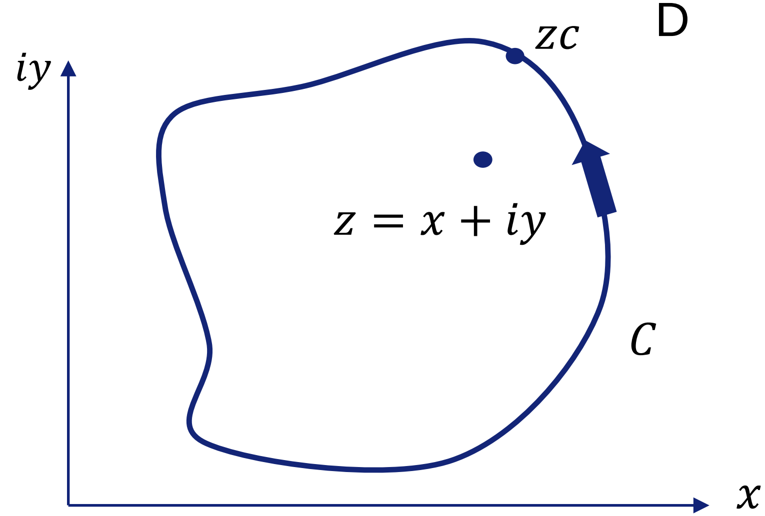
THE TOOL: CAUCHY INTEGRAL FORMULA

$b(z)$= $b_y(\mathrm{z}) + \mathrm{i}b_x(z)$ analytic in domain D simply connected and C an oriented closed path in D

$$b(z) = \frac{1}{2\pi i}\int_C \frac{b(z_c)}{z_c - z} dz_c$$

Cauchy integral formula

In other words, if we know $b(z)$ on the contour C, $b(z)$ can be reconstructed at any point inside C.



(be careful with orientation of C around z formula for C oriented positively)

Also: $$\int_C b(z)\, dz = 0$$ (Cauchy' theorem)

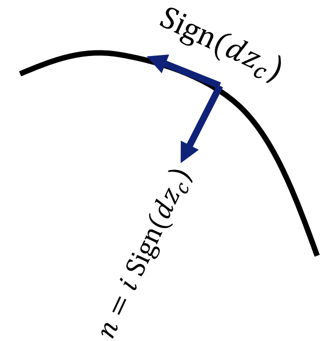
CAUCHY INTEGRAL & BIOT AND SAVART

Some little transformation & interpretation

$$b(z) = \frac{1}{2\pi i}\int_C \frac{b(z_c)}{z_c - z}\, dz_c$$

$$b(z) = \frac{1}{2\pi}\int_C \frac{b(z_c)\overbrace{i\ \mathrm{Sign}(dz_c)}^{n(z_c)}}{z - z_c}\, |dz_c|$$

$$Sign(z) = \frac{z}{|z|}$$

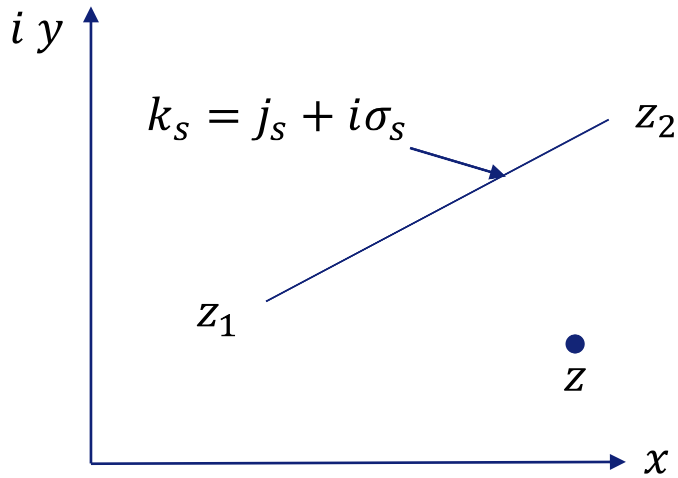


$$b\,n = (b_y + i b_x)\,(n_x + i n_y)$$
$$= j_s + i\sigma_s = k_s$$

$$b(z) = \frac{1}{2\pi}\int_C \frac{k_s}{z - z_c}\, |dz_c|$$

2D Biot & Savart law

CONSTANT SURFACE DISTRIBUTION



$$Sign(z) = \frac{z}{|z|}$$

Potential $$\mathcal{A}(z, z_1, z_2) = a + iv = k_s\,(u_2 - u_1 - u_2 \ln(u_2) + u_1 \ln(u_1))$$

Field $$\mathcal{B}(z, z_1, z_2) = b_y + ib_x = \frac{k_s}{Sign(z_2 - z_1)} \ln \frac{u_1}{u_2}$$

$$u_1 = \frac{z - z_1}{Sign(z_2 - z_1)}$$

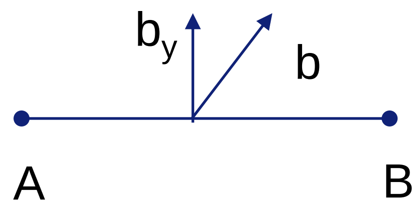
Order n gradient
1= quadrupole term

$$\mathcal{G}(z, z_1, z_2, n) = \frac{k_s (n-1)!\ (-1)^{n-1}}{Sign(z_2 - z_1)} \left(\frac{1}{(z - z_1)^n} - \frac{1}{(z - z_2)^n}\right)$$

$$u_2 = \frac{z - z_2}{Sign(z_2 - z_1)}$$

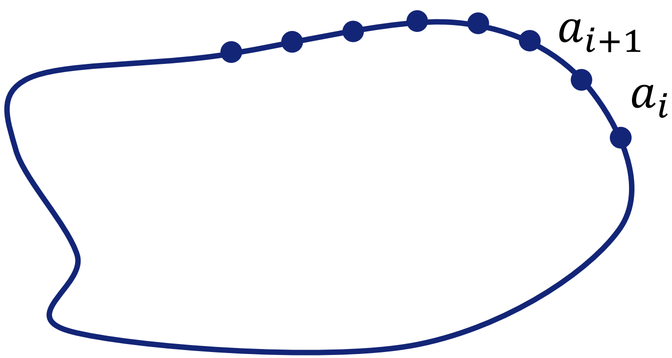
STRETCHED WIRE

Wire moving from A to B



$$\emptyset_{AB} = \int_A^B b_y dx = \int_A^B \frac{\partial a}{\partial x} dx = a(B) - a(A)$$

We only deal the real part of the complex potential $\mathcal{A}$



Closed boundary with N segments

Potential a known at the middle of each segment using stretched wire measurements.

Focus: determine a surface current distribution on the boundary generating the potential a

SURFACE CURRENT AT BOUNDARY

$$A = M\,J_s$$

A vector with potential $a_i, i = (1, N)$

J_s vector with surface current J_{si} , $i = (1, N)$

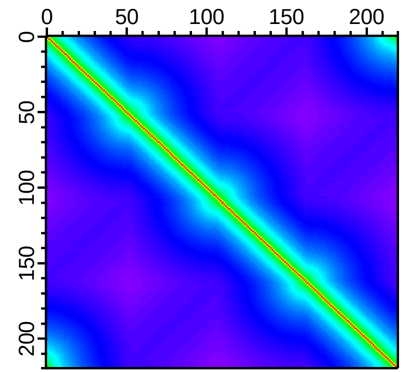
M square $N\ x\ N$ matrix with terms

$$m_{ij} = Re(\mathcal{A}\left((z_i + z_{i+1})/2, z_j, z_{j+1}\right)$$

Segment i delimited by z_i and z_{i+1}

M is a (symmetric) matrix which can be inverted

$$J_s = M^{-1} A$$



Example of matrix M

FIELD CALCULATION

With j_s known on boundary C

field $b\,(z) = \sum_{i=1}^{N} j_{si}\mathcal{B}(z, z_i, z_{i+1})$

potential $a\,(z) = \sum_{i=1}^{N} j_{si}\,\mathcal{A}(z, z_i, z_{i+1})$

gradient $g\,(z, n) = \sum_{i=1}^{N} j_{si}\mathcal{G}(z, z_i, z_{i+1}, n)$

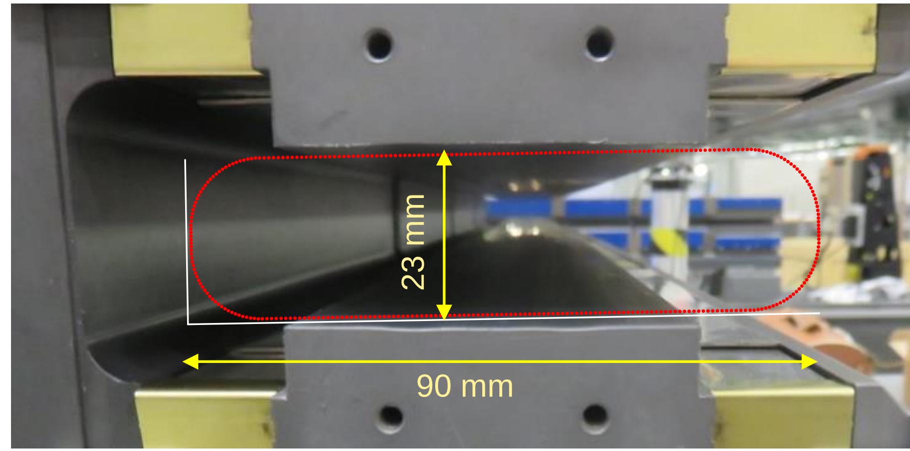
For any point inside boundary

Outside boundary: field is not diverging but do not represent correctly the magnet

APPLICATION TO MAGNETIC MEASUREMENTS

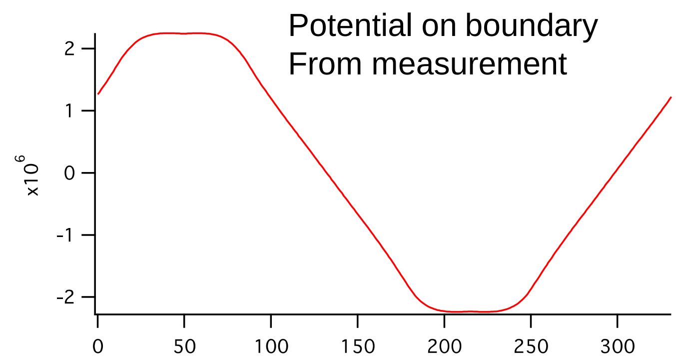
- Synchronized Newport axes
- XPS controller
- Carbon fiber wire
- Keythley 2182 nano-voltmeter
- Wire moved along a **closed** boundary
- Constant speed on trajectory ~ 20 mm/s
- Voltage integrated over each segment

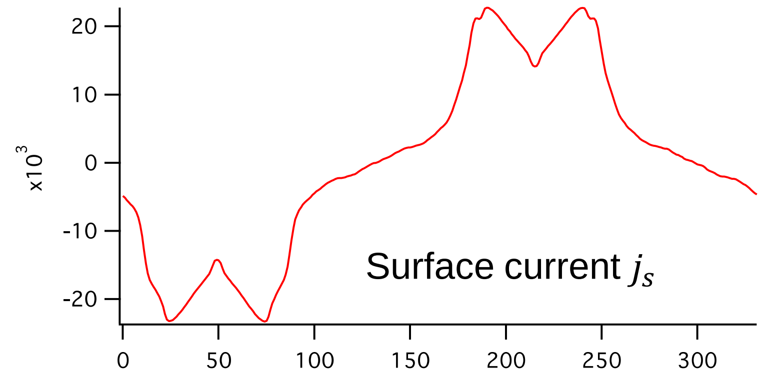
DLs

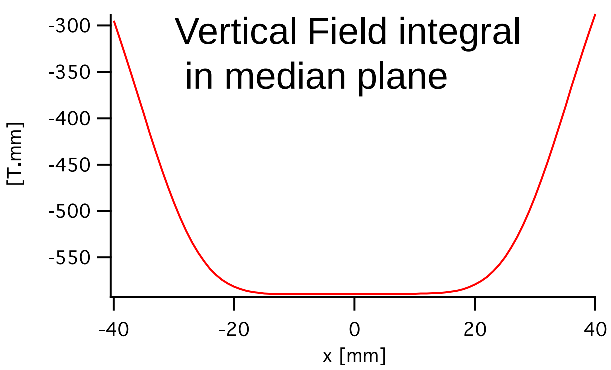


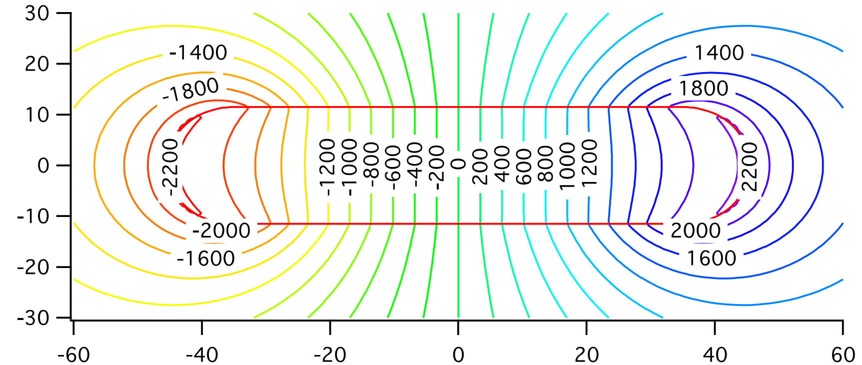
- 332 segments
- pole gap 24.5 mm
- Segment length 0.7 mm

DLS (DIPOLE)



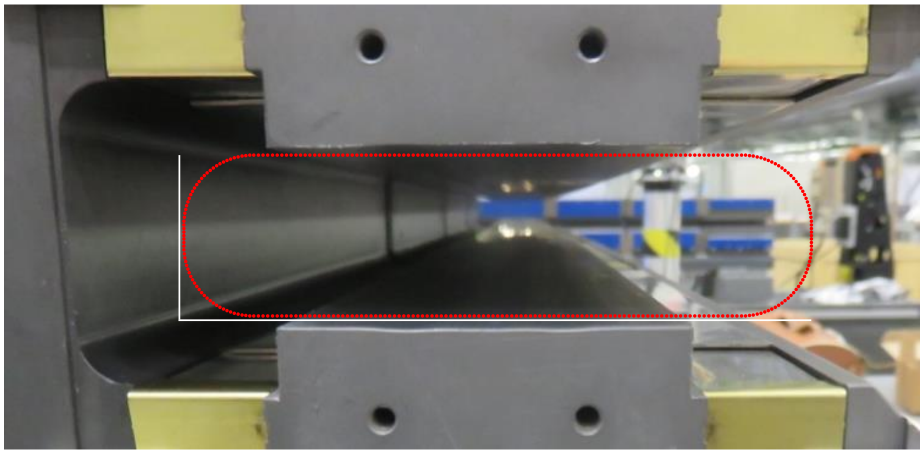


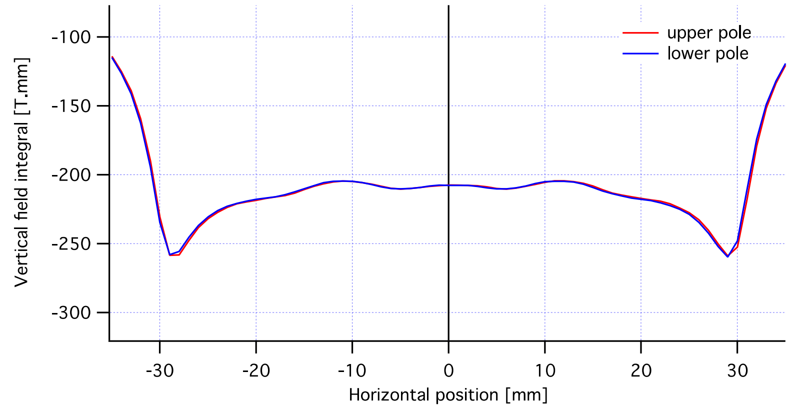




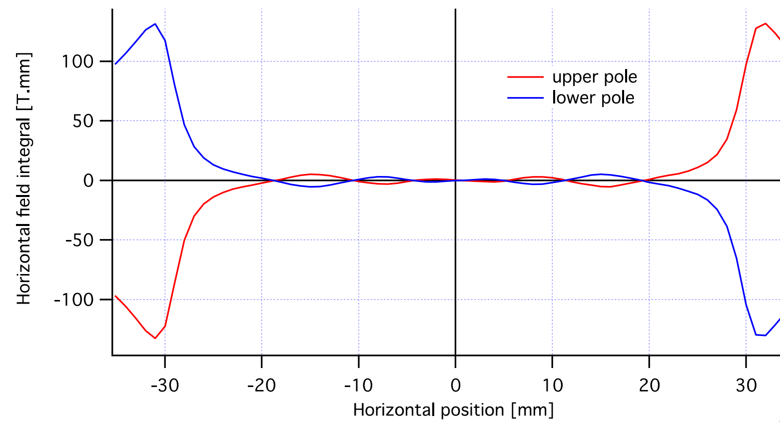
Potential inside boundary (and outside boundary)

SOME COOL STUFF





Looking at pole details
~ 1.25 mm from pole

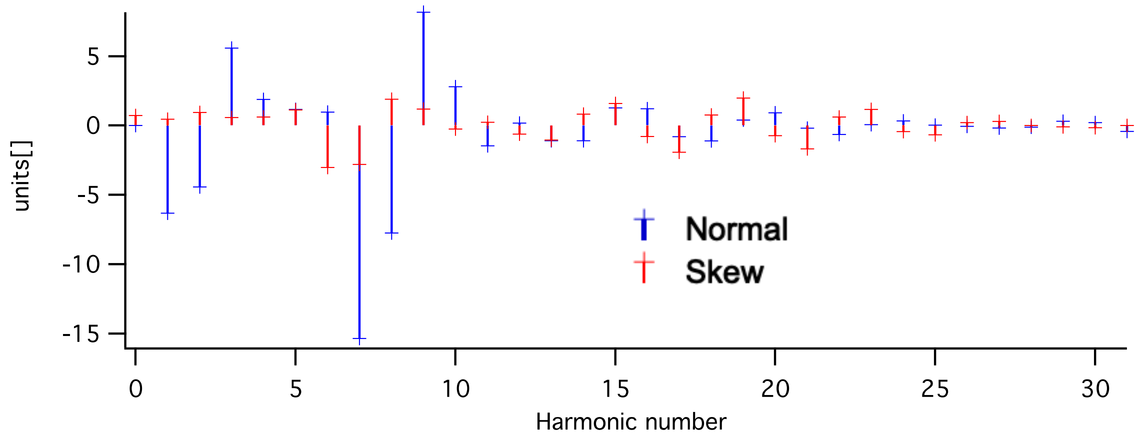


HARMONIC ANALYSIS

Can be done directly at any point inside contour

Circle of radius r and center z_0
Circle should be inside contour

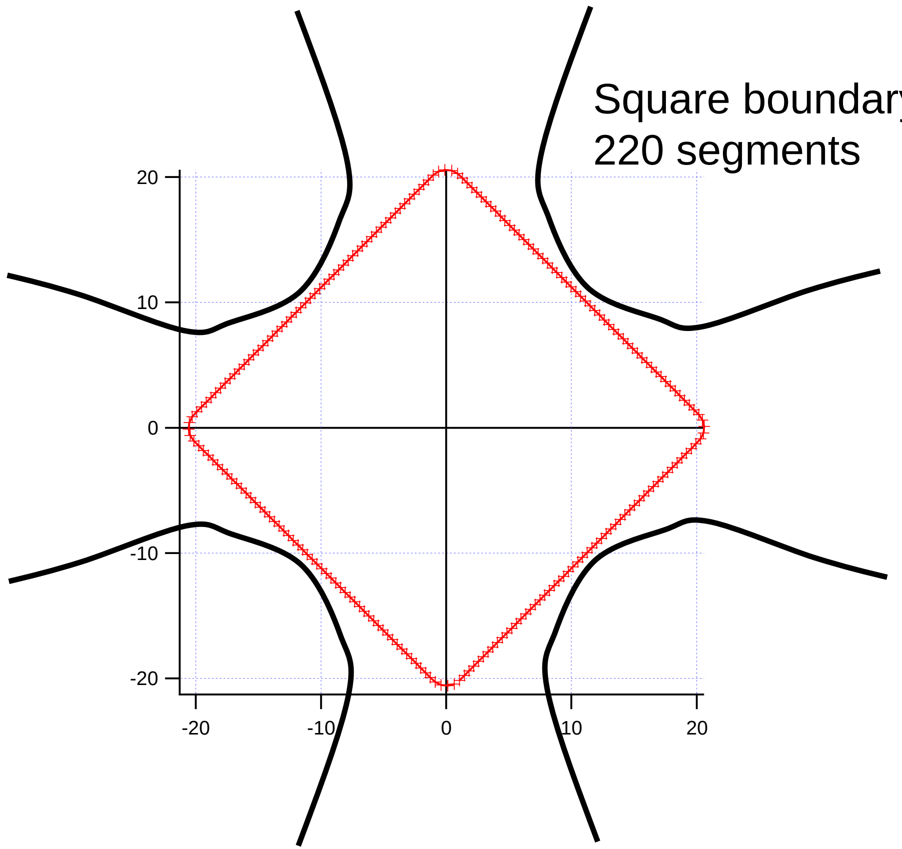
$$\text{Hn}= g\,(z_0, \text{n})r^n$$



Example with measurements
z_0= center of dipole
(normalization to dipole)

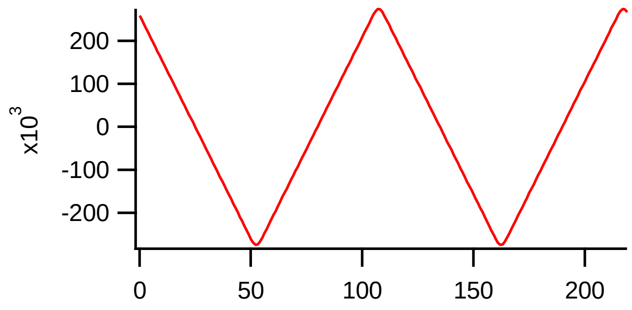
EXAMPLE2: QUADRUPOLE

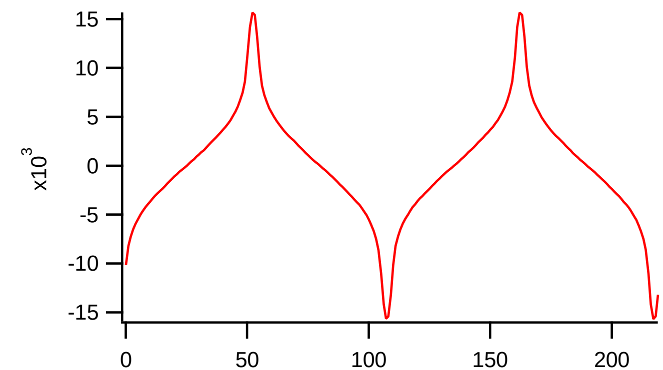
Square boundary
220 segments



Measured quadrupole: QD2

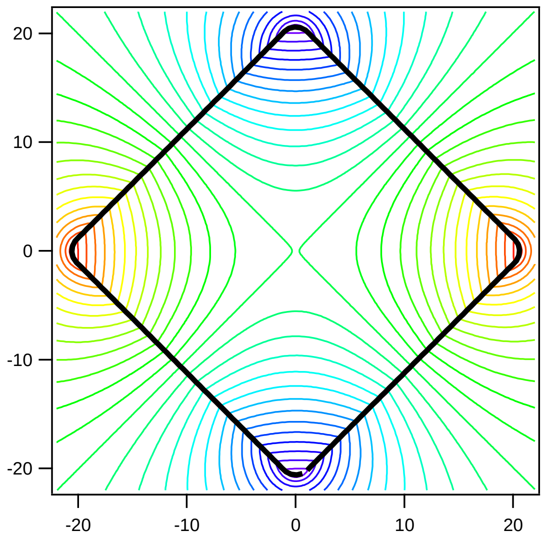
Potential on boundary



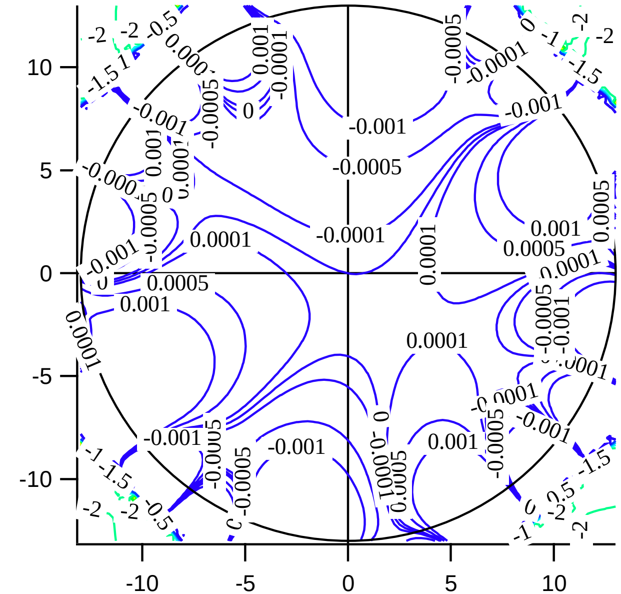


Surface current j_s

EXAMPLE 2: QUADRUPOLE

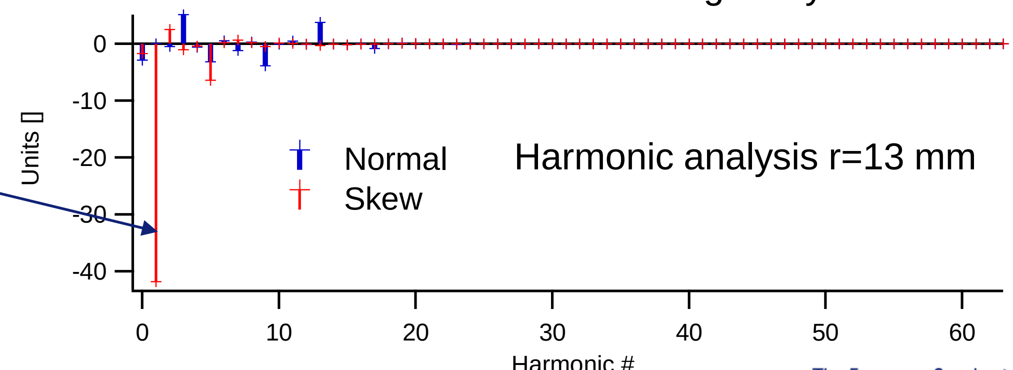


Potential inside (and outside boundary)



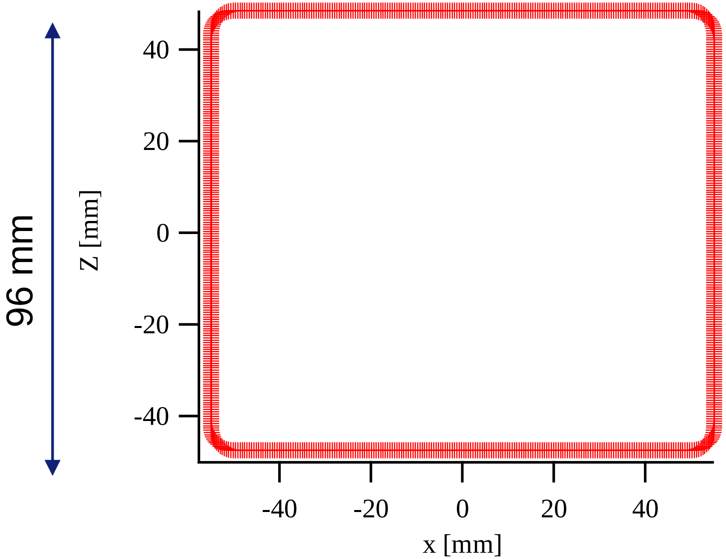
Gradient homogeneity

Magnet rotation not corrected

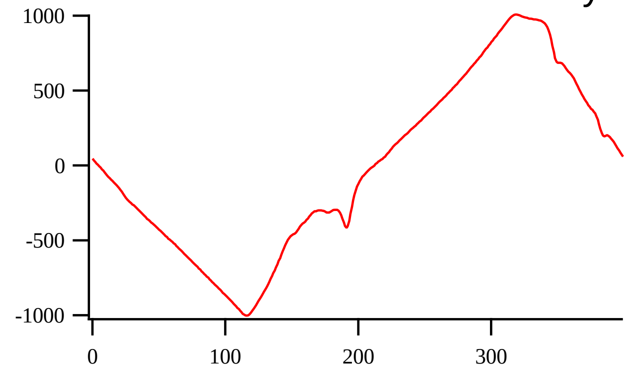


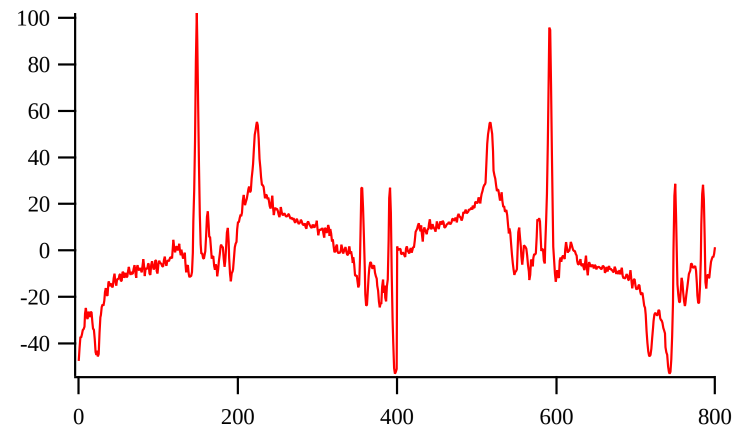
EXAMPLE 3: UNDULATOR

Undulator gap 100 mm
400 segments



Potential on boundary

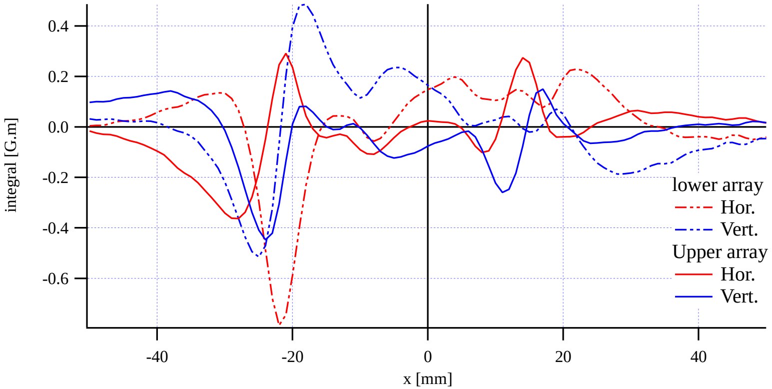




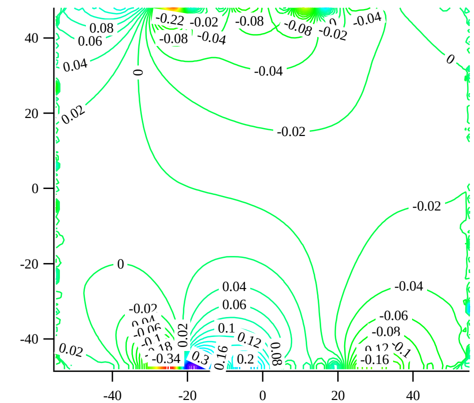
Surface current j_s

EXAMPLE 3: UNDULATOR

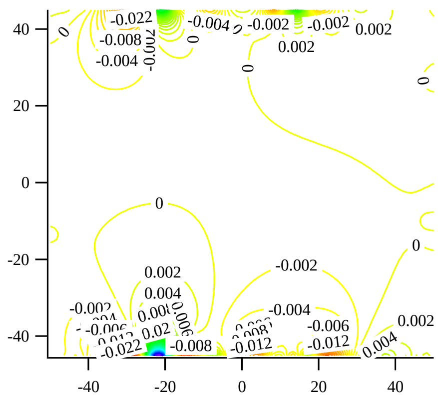
Example of computation



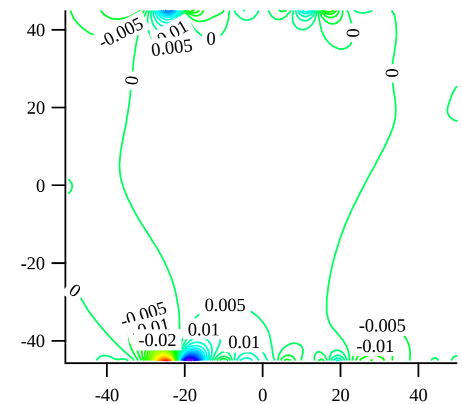
At 2 mm from magnet surface



Vertical field integral in gap [G.m]



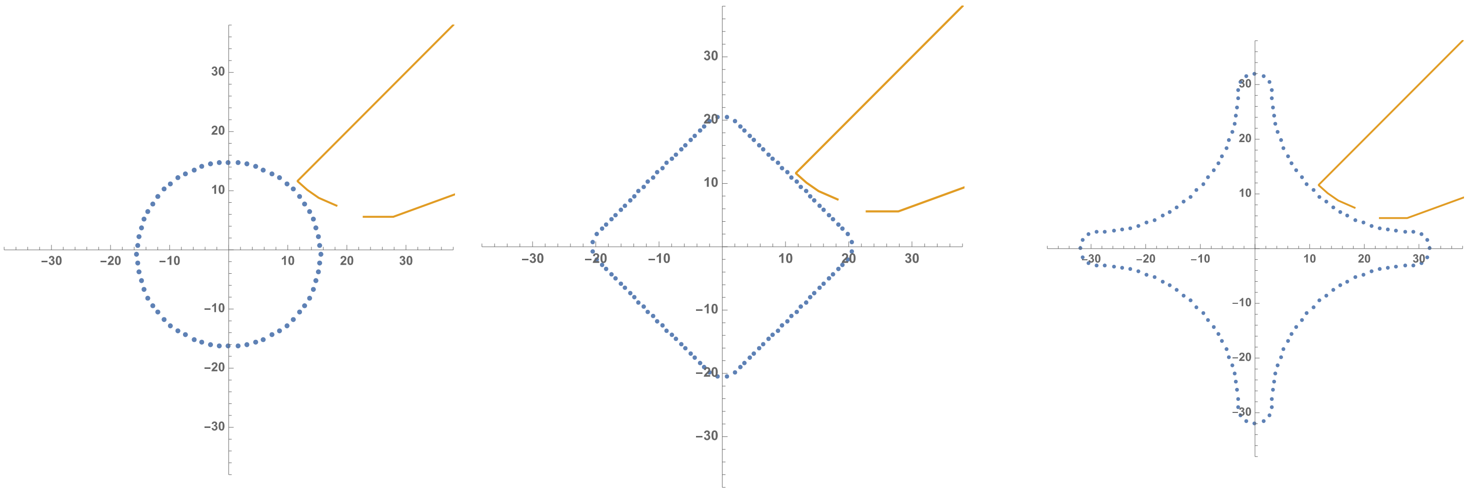
Normal quadrupole



skew quadrupole

DEVELOPMENTS

- Efficient path descriptor



- Additional building elements
  - Portion of arcs (analytical formulation done)
  - Polynomial surface distribution

SUMMARY

- A new method for integrated field measurement developed
- Well adapted for stretched wire
- Method was used for the measurement of all DLs
- Presently in routine use for ID field integral measurements @ ESRF
  - Seems interesting for small gap devices

THANK YOU

IMMW21